

D0-brane realizations of the resolution of a reduced singular curve

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Abstract

Based on examples from superstring/D-brane theory since the work of Douglas and Moore on resolution of singularities of a superstring target-space Y via a D-brane probe, the richness and the complexity of the stack of punctual D0-branes on a variety, and as a guiding question, we lay down a conjecture that any resolution $Y' \rightarrow Y$ of a variety Y over \mathbb{C} can be factored through an embedding of Y' into the stack $\mathfrak{M}_r^{0_{\text{As},f}}(Y)$ of punctual D0-branes of rank r on Y for $r \geq r_0$ in \mathbb{N} , where r_0 depends on the germ of singularities of Y . We prove that this conjecture holds for the resolution $\rho : C' \rightarrow C$ of a reduced singular curve C over \mathbb{C} . In string-theoretical language, this says that the resolution C' of a singular curve C always arises from an appropriate D0-brane aggregation on C and that the rank of the Chan-Paton module of the D0-branes involved can be chosen to be arbitrarily large.

Key words: D-brane, resolution, singularity; punctual D0-brane, stack; singular curve, normalization; embedding, separation of points, separation of tangents.

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0. Introduction and outline.

The work [D-M] of Michael Douglas and Gregory Moore on resolution of singularities of a superstring target-space Y via a D-brane probe (i.e., the realization of a resolution Y' of Y as a space of vacua – namely, a moduli space in quantum-field-theoretical sense – of the world-volume quantum field theory of the D-brane probe) has influenced many studies both on the mathematics and the string-theory side. (See also a related work [J-M] of Clifford Johnson and Robert Myers.) The attempt to understand the underlying geometry behind the setup of [D-M] is indeed part of the driving force that leads us to the current setting of D-branes in the project (cf. [L-Y1] and [L-Y2]). Based on examples¹ from superstring/D-brane theory since [D-M], the richness and the complexity of the stack $\mathfrak{M}_p^{0_{\text{Az}}^f}(Y)$ of punctual D0-branes on a variety Y , and as a guiding question, we lay down in this note² a conjecture that any resolution $Y' \rightarrow Y$ of a variety Y over \mathbb{C} can be factored through an embedding of Y' into the stack $\mathfrak{M}_r^{0_{\text{Az}}^f}(Y)$ of punctual D0-branes of rank r on Y for $r \geq r_0$ in \mathbb{N} , where r_0 depends on the germ of singularities of Y ; cf. Sec. 1. For the one-dimensional case, we prove that this conjecture holds for the resolution $\rho: C' \rightarrow C$ of a reduced singular curve C over \mathbb{C} ; cf. Sec. 2. In string-theoretical language, this says that the resolution C' of a singular curve C always arises from an appropriate D0-brane aggregation on C and that the rank of the Chan-Paton module of the D0-branes involved can be chosen to be arbitrarily large.

Remark 0.1. [another aspect]. It should be noted that there is another direction of D-brane resolutions of singularities (e.g. [As], [Br], [Ch]), from the point of view of (hard/massive/solitonic) D-branes (or more precisely B-branes) as objects in the bounded derived category of coherent sheaves. Conceptually that aspect and ours (for which D-branes are soft in terms of string tension) are in different regimes of a refined Wilson’s theory-space of $d = 2$ supersymmetric field theory-with-boundary on the open-string world-sheet.³ Being so, there should be an interpolation between these two aspects. It would be very interesting to understand such details.

Convention. Standard notations, terminology, operations, facts in (1) algebraic geometry; (2) coherent sheaves; (3) resolution of singularities; (4) stacks can be found respectively in (1) [Ha]; (2) [H-L]; (3) [Hi], [Ko]; (4) [L-MB].

- All varieties, schemes and their products are over \mathbb{C} ; a ‘*curve*’ means a 1-dimensional proper scheme over \mathbb{C} ; a ‘*stack*’ means an *Artin stack*.
- The ‘*support*’ $\text{Supp}(\mathcal{F})$ of a coherent sheaf \mathcal{F} on a scheme Y means the *scheme-theoretical support* of \mathcal{F} ; \mathcal{I}_Z denotes the *ideal sheaf* of a subscheme of Z of a scheme Y .
- The current note continues the study in [L-Y1] (arXiv:0709.1515 [math.AG], D(1)), [L-Y2] (arXiv:0901.0342 [math.AG], D(3)), and [L-Y3] (arXiv:0907.0268 [math.AG], D(4)) with some background from [L-L-S-Y] (arXiv:0809.2121 [math.AG], (2)). A partial review of D-branes and Azumaya noncommutative geometry is given in [L-Y4] (arXiv:1003.1178 [math.SG], D(6)). Notations and conventions follow these early works when applicable.

¹Unfamiliar readers are highly recommended to use keyword search to get a taste of the vast literature.

²In part, for a subsection of a talk under the title ‘*Azumaya noncommutative geometry and D-branes - an origin of the master nature of D-branes*’ to be delivered in the workshop *Noncommutative algebraic geometry and D-branes*, December 12 – 16, 2011, organized by Charlie Beil, Michael Douglas, and Peng Gao, at Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY.

³*For mathematicians:* See [W-K] for the origin of the notion of Wilson’s theory-space and, for example, [H-I-V] and [H-H-P] for the case of $d = 2$ supersymmetric quantum field theories with boundary.

Outline.

0. Introduction.
1. The stack of punctual D0-branes on a variety and an abundance conjecture.
 - D-branes as morphisms from Azumaya noncommutative spaces with a fundamental module.
 - The stack of punctual D0-branes on a variety and an abundance conjecture.
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 - Separation of points in $\rho^{-1}(p)$ via punctual D0-branes at p .
 - Construction of embeddings $C' \hookrightarrow \mathfrak{M}_p^{0, \text{Az}^f}(C)$ that descend to ρ .

1 The stack of punctual D0-branes on a variety and an abundance conjecture.

We collect a few most essential definitions and setups for this sub-line of the project. Readers are referred to [L-Y4] for a more thorough review of the first part of the project and stringy-theoretical remarks on how inputs from [Po1], [Po2], and [Wi] lead to such a setting.

D-branes as morphisms from Azumaya noncommutative spaces with a fundamental module.

Our starting point is the following prototypical definition of D-branes that comes from a mathematical understanding of [Po1], [Po2] from Joseph Polchinski and [Wi] from Edward Witten based on how Alexandre Grothendieck developed the theory of schemes in modern (commutative) algebraic geometry:

Definition 1.1. [D-brane]. Let Y be a variety (over \mathbb{C}). A *D-brane* on Y is a *morphism* φ from an Azumaya noncommutative space-with-a-fundamental-module $(X^{\text{Az}}, \mathcal{E}) := (X, \mathcal{O}_X^{\text{Az}}, \mathcal{E})$ to Y . Here, X is a scheme over \mathbb{C} , \mathcal{E} a locally free \mathcal{O}_X -module, and $\mathcal{O}_X^{\text{Az}} = \text{End}_{\mathcal{O}_X}(\mathcal{E})$; and φ is defined through *an equivalence class of gluing systems of ring homomorphisms* given by $\varphi^\# : \mathcal{O}_Y \rightarrow \mathcal{O}_X^{\text{Az}}$. The rank of \mathcal{E} is called the *rank* of the D-brane.

Similar to the fact that the data of a morphism $f : X \rightarrow Y$ between schemes can be encoded completely by its graph Γ_f as a subscheme in $X \times Y$, the data of φ is also encoded completely by its graph Γ_φ :

Definition 1.2. [φ in terms of its graph Γ_φ]. The *graph* of a morphism in Definition 1.1 is given by an $\mathcal{O}_{X \times Y}$ -module $\tilde{\mathcal{E}}$ that is flat over X and of relative dimension 0. In detail, let $pr_1 : X \times Y \rightarrow X$, $pr_2 : X \times Y \rightarrow Y$ be the projection map, and $f_\varphi : \text{Supp}(\tilde{\mathcal{E}}) \rightarrow Y$ be the restriction of pr_2 . Then $\tilde{\mathcal{E}}$ defines a morphism φ in Definition 1.1 as follows:

- $\mathcal{E} = pr_{1*} \tilde{\mathcal{E}}$;
- note that $\text{Supp}(\tilde{\mathcal{E}})$ is affine over X ; thus, the gluing system of ring homomorphisms $f_\varphi^\# : \mathcal{O}_Y \rightarrow \mathcal{O}_{\text{Supp}(\tilde{\mathcal{E}})}$ defines a gluing system of ring-homomorphisms $\varphi^\# : \mathcal{O}_Y \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{E}) = \mathcal{O}_X^{\text{Az}}$, which defines φ .

It is worth emphasizing that, *unlike* the standard setting for a morphism between ringed topological spaces in commutative geometry, in general φ specifies only a correspondence from X to Y via the diagram

$$\begin{array}{ccc} X_\varphi := \text{Supp}(\tilde{\mathcal{E}}) & \xrightarrow{f_\varphi} & Y \\ \pi_\varphi \downarrow & & \\ X & & \end{array},$$

not a morphism from X to Y .

Definition 1.2 suggests another equivalent description of φ .

Definition 1.3. [φ as morphism to stack of D0-branes]. Let $\mathfrak{M}^{0\text{Az}^f}(Y)$ be the stack of 0-dimensional \mathcal{O}_Y -modules. It follows from Definition 1.2 that this is precisely the stack of D0-branes on Y in the sense of Definition 1.1 and, hence, the notation. Then, a morphism φ in Definition 1.1 is specified by a morphism $\hat{\varphi} : X \rightarrow \mathfrak{M}^{0\text{Az}^f}(Y)$.

The stack of punctual D0-branes on a variety and an abundance conjecture.

Definition 1.4. [stack of punctual D0-branes]. Let Y be a variety. By a *punctual* 0-dimensional \mathcal{O}_Y -module, we mean a 0-dimensional \mathcal{O}_Y -module \mathcal{F} whose $\text{Supp}(\mathcal{F})$ is a single point (with structure sheaf an Artin local ring). By Definition 1.2, \mathcal{F} specifies a D0-brane on Y , which is called a *punctual D0-brane*. It is a morphism from an Azumaya point with a fundamental module to Y that takes the fundamental module to a punctual 0-dimensional \mathcal{O}_Y -module. Let $\mathfrak{M}_r^{0\text{Az}^f/p}(Y)$ be the *stack of punctual D0-branes of rank r on a variety Y* . It has an Artin stack with atlas constructed from Quot-schemes. There is a morphism $\pi_Y : \mathfrak{M}_r^{0\text{Az}^f/p}(Y) \rightarrow Y$ that takes \mathcal{F} to $\text{Supp}(\mathcal{F})$ with the reduced scheme structure. π_Y is essentially the Hilbert-Chow/Quot-Chow morphism.

The following two conjectures are motivated by the various examples in string theory concerning D-brane resolution of singularities of a superstring target-space and the richness and the complexity of the stack $\mathfrak{M}_r^{0\text{Az}^f/p}(Y)$:

Conjecture 1.5. [D0-brane resolution of singularity]. Any resolution $Y' \rightarrow Y$ of a variety Y can be factored through an embedding of Y' into the stack $\mathfrak{M}_r^{0\text{Az}^f/p}(Y)$ of punctual D0-branes of rank r on Y for any $r \geq r_0$ in \mathbb{N} , where r_0 depends only on the germ of singularities of Y .

Conjecture 1.5 is a weaker form of the following stronger form of an abundance conjecture:

Conjecture 1.6. [abundance]. Any birational morphism $Y' \rightarrow Y$ between varieties over \mathbb{C} can be factored through an embedding of Y' into the stack $\mathfrak{M}_r^{0\text{Az}^f/p}(Y)$ of punctual D0-branes of rank r on Y for any $r \geq r_0$ in \mathbb{N} , where r_0 depends only on the germ of singularities of Y and the germ of singularities of Y' .

This says that all the birational models of and over Y are already contained in the stack $\mathfrak{M}_r^{0\text{Az}^f/p}(Y)$ of punctual D0-branes on Y . All the birational transitions between birational models of and over Y happens as correspondences inside $\mathfrak{M}_r^{0\text{Az}^f/p}(Y)$ (and hence the name of the conjecture) – an intrinsic stack over Y , locally of finite type, that is canonically associated to Y .

Remark 1.7. [*string-theoretical remark*]. A standard setting (cf. [D-M]) in D-brane resolution of singularities of a (complex) variety Y (which is a singular Calabi-Yau space in the context of string theory) is to consider a super-string target-space-time of the form $\mathbb{R}^{(9-2d)+1} \times Y$ and an (effective-space-time-filling) $D(9-2d)$ -brane whose world-volume sits in the target space-time as a submanifold of the form $\mathbb{R}^{(9-2d)+1} \times \{p\}$. Here, d is the complex dimension of the variety Y and $p \in Y$ is an isolated singularity of Y . When considering only the geometry of the internal part of this setting, one sees only a D0-brane on Y . This explains the role of D0-branes in the statement of Conjecture 1.5 and Conjecture 1.6. In the physics side, the exact dimension of the D-brane (rather than just the internal part) matters since supersymmetries and their superfield representations in different dimensions are not the same and, hence, dimension does play a role in writing down a supersymmetric quantum-field-theory action for the world-volume of the $D(9-2d)$ -brane probe. In the above mathematical abstraction, these data are now reflected into the richness, complexity, and a master nature of the stack $\mathfrak{M}_r^{0_{Az^f}}(Y)$ that is intrinsically associated to the internal geometry. The precise dimension of the D-brane as an object sitting in or mapped to the whole space-time becomes irrelevant.

2 Realizations of resolution of singular curves via D0-branes.

Let C be a reduced singular curve over \mathbb{C} and

$$\rho : C' \longrightarrow C$$

be the resolution of singularities of C . In the current 1-dimensional case, the singularities of C are isolated and ρ is realized by the normalization of C . In particular, ρ is an affine morphism. The built-in \mathcal{O}_C -module homomorphism $\rho^\sharp : \mathcal{O}_C \rightarrow \rho_* \mathcal{O}_{C'}$ determines a subsheaf $\mathcal{A}_C \subset \mathcal{O}_{C'}$ of \mathbb{C} -subalgebras with the induced morphism $C' \rightarrow \mathbf{Spec} \mathcal{A}_C$ identical to ρ . Let $p' \in C'$ be a closed point, $p := \rho(p')$, and $\mathfrak{m}_{p'} = (t)$ (resp. \mathfrak{m}_p) be the maximal ideal of $\mathcal{O}_{C',p'}$ (resp. $\mathcal{O}_{C,p}$). Then $\rho^\sharp(\mathfrak{m}_p) \cdot \mathcal{O}_{C',p'} = (t^{n_{p'}})$ for some $n_{p'} \in \mathbb{N}$. $n_{p'} > 1$ if and only if $p \in C_{\text{sing}} :=$ the singular locus of C . We show in this section that:

Proposition 2.1. [**one-dimensional case**]. *Conjecture 1.5 holds for $\rho : C' \rightarrow C$. Namely, there exists an $r_0 \in \mathbb{N}$ depending only on the tuple $(n_{p'})_{\rho(p') \in C_{\text{sing}}}$ and a (possibly empty) set $\{b.i.i.(p) : p \in C_{\text{sing}}, C \text{ has multiple branches at } p\}$ (cf. Definition 2.6), both associated to the germ of C_{sing} in C , such that, for any $r \geq r_0$, there exists an embedding $\tilde{\rho} : C' \hookrightarrow \mathfrak{M}_r^{0_{Az^f}}(C)$ that makes the following diagram commute:*

$$\begin{array}{ccc} & & \mathfrak{M}_r^{0_{Az^f}}(C) \\ & \nearrow \tilde{\rho} & \downarrow \pi_C \\ C' & \xrightarrow{\rho} & C \end{array} \quad .$$

Basic setup and a criterion for nontrivial extensions of modules.

Consider the induced affine morphism $\text{id}_{C'} \times \rho : C' \times C' \rightarrow C' \times C$. Let $pr'_1 : C' \times C' \rightarrow C'$ (the first factor), $pr'_2 : C' \times C' \rightarrow C'$ (the second factor), $pr_1 : C' \times C \rightarrow C'$, and $pr_2 : C' \times C \rightarrow C$ be the projection maps. Let $\tilde{\mathcal{E}}'$ be a coherent sheaf on $C' \times C'$ that is flat over C' under pr'_1 , with support in an infinitesimal neighborhood of the diagonal $\Delta_{C'} \subset C' \times C'$. Then $\tilde{\mathcal{E}} := (\text{id}_{C'} \times \rho)_*(\tilde{\mathcal{E}}')$ is a coherent sheaf on $C' \times C$ that is flat over C' under pr_1 , with support in an infinitesimal neighborhood of the graph Γ_ρ of ρ in $C' \times C$.

Lemma 2.2. [commutativity of push-forward and restriction]. *Let $p' \in C'$ be a closed point. Then $(id_{C'} \times \rho)_*(\tilde{\mathcal{E}}'|_{\{p'\} \times C'}) = \tilde{\mathcal{E}}|_{\{p'\} \times C}$.*

Proof. As $\tilde{\mathcal{E}}'$ is flat over C' under pr'_1 , one has the exact sequence

$$0 \longrightarrow \mathcal{I}_{\{p'\} \times C'} \otimes_{\mathcal{O}_{C'}} \tilde{\mathcal{E}}' \longrightarrow \tilde{\mathcal{E}}' \longrightarrow \tilde{\mathcal{E}}'|_{\{p'\} \times C'} \longrightarrow 0.$$

Since $id_{C'} \times \rho$ is affine, $(id_{C'} \times \rho)_* : \text{Coh}(C') \rightarrow \text{Coh}(C)$ is exact and one has

$$\begin{array}{ccccccc} 0 & \longrightarrow & (id_{C'} \times \rho)_*(\mathcal{I}_{\{p'\} \times C'} \otimes_{\mathcal{O}_{C'}} \tilde{\mathcal{E}}') & \longrightarrow & \tilde{\mathcal{E}} & \longrightarrow & (id_{C'} \times \rho)_*(\tilde{\mathcal{E}}'|_{\{p'\} \times C'}) \longrightarrow 0 \\ & & \parallel & & & & \\ & & \mathcal{I}_{\{p'\} \times C} \otimes_{\mathcal{O}_C} \tilde{\mathcal{E}} & & & & \end{array},$$

where the top horizontal line is an exact sequence. This proves the lemma. \square

Remark/Notation 2.3. [general restriction over a base]. Lemma 2.2 holds more generally with p' replaced by a subscheme of C' , by the same proof with the replacement. We'll denote the restriction of a coherent sheaf $\tilde{\mathcal{F}}'$ (resp. $\tilde{\mathcal{F}}$) on $C' \times C'$ (resp. $C' \times C$) over a subscheme Z' of the base C' by $\tilde{\mathcal{F}}'_{Z'}$ (resp. $\tilde{\mathcal{F}}_{Z'}$).

Let $v_{p'} \simeq \text{Spec}(\mathbb{C}[\varepsilon])$, where $\varepsilon^2 = 0$, be the subscheme of the base C' that corresponds to the \mathbb{C} -algebra quotient $\mathcal{O}_{C',p'} \rightarrow \mathbb{C}[\varepsilon]$ with $t \mapsto \varepsilon$. Then the restriction of $\tilde{\mathcal{E}}'$ over $v_{p'}$ determines an element $\alpha'_{p'} \in \text{Ext}^1_{C'}(\tilde{\mathcal{E}}'_{p'}, \tilde{\mathcal{E}}'_{p'})$. Similarly, the restriction of \mathcal{E} over $v_{p'}$ determines an element $\alpha_{p'} =: \rho_* \alpha'_{p'} \in \text{Ext}^1_C(\tilde{\mathcal{E}}_{p'}, \tilde{\mathcal{E}}_{p'})$. Let $p := \rho(p')$ and recall $t \in \mathcal{O}_{C',p'}$ and $n_{p'} \in \mathbb{N}$ from the beginning of this section. Let us first state an elementary criterion for non-splitability of a short exact sequence, whose proof is immediate:

Lemma 2.4. [criterion of non-splitability]. *Let W be a scheme and*

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}_1 \longrightarrow 0$$

be an exact sequence of \mathcal{O}_W -modules that represents a class $\beta \in \text{Ext}^1_W(\mathcal{F}_1, \mathcal{F}_2)$. Suppose that there exist a point $w \in W$ and a local function $f \in \mathcal{O}_{W,w}$ such that, for the associated $\mathcal{O}_{W,w}$ -modules (still denoted the same), $f \cdot \mathcal{F}_1 = f \cdot \mathcal{F}_2 = 0$ while $f \cdot \mathcal{G} \neq 0$. Then, $\beta \neq 0$; namely, the above sequence doesn't split.

Corollary 2.5. [push-forward of jet]. *Continuing the main-line discussions and notations. Let α' be given by the exact sequence*

$$0 \longrightarrow \tilde{\mathcal{E}}'_{p'} \longrightarrow \tilde{\mathcal{F}}' \xrightarrow{j} \tilde{\mathcal{E}}'_{p'} \longrightarrow 0$$

of $\mathcal{O}_{C'}$ -modules. Denote the same for the associated exact sequence of $\mathcal{O}_{C',p'}$ -modules. As such, suppose that there is an $l \in \mathbb{N}$ such that $(t^{n_{p'}})^l \cdot \tilde{\mathcal{E}}' = 0$ while $(t^{n_{p'}})^{l+1} \cdot \tilde{\mathcal{F}}' \neq 0$. Then $\alpha \neq 0$ in $\text{Ext}^1_C(\tilde{\mathcal{E}}_{p'}, \tilde{\mathcal{E}}_{p'})$.

Proof. Note that the multiplication of t by an invertible element in $\mathcal{O}_{C',p'}$ (i.e. by an element in $\mathcal{O}_{C',p'} - \mathfrak{m}_{p'}$) won't alter its nilpotency behavior on the modules in question. The corollary follows immediately from Lemma 2.4 and the observation that, up to a multiplication by an invertible element in $\mathcal{O}_{C',p'}$, one may assume that $t^{n_{p'}} \in \rho^\sharp(\mathcal{O}_{C,p})$. \square

Separation of points in $\rho^{-1}(p)$ via punctual D0-branes at p .

Let $p \in C_{\text{sing}}$ and \hat{C} be the formal neighborhood (as an ind-scheme) of p in C . Then each irreducible component \hat{C}_i , $i = 1, \dots, k$, of \hat{C} corresponds to a branch of the germ of p in C . Assume that $k \geq 2$. Then the intersection of two distinct components \hat{C}_i and \hat{C}_j of \hat{C} is represented by a punctual 0-dimensional subscheme $Z_{ij} = Z_{ji}$ of C at p of finite length $l_{ij} = l_{ji}$.

Definition 2.6. [branch intersection index]. For $k \geq 2$, define the *branch intersection index* $b.i.i.(p)$ at $p \in C_{\text{sing}}$ to be

$$b.i.i.(p) := \max\{l_{ij} : 1 \leq i, j \leq k; i \neq j\}.$$

Let $p \in C_{\text{sing}}$, $\rho^{-1}(p) = \{p'_1, \dots, p'_k\}$, and \hat{C}'_i be the formal neighborhood of p'_i in C' . Then $\rho : C' \rightarrow C$ induces a morphism $\hat{\rho}_i : \hat{C}'_i \rightarrow \hat{C}$ of ind-schemes, for $i = 1, \dots, k$. The image $\hat{\rho}_i(\hat{C}'_i)$ is a branch of \hat{C} , which we may assume to be \hat{C}_i , after relabeling, since different \hat{C}'_i 's are mapped to different branches of \hat{C} under $\hat{\rho}_i$. Let $\mathfrak{m}_{p'_i} = (u_i)$ be the maximal ideal of \mathcal{O}_{C', p'_i} ;

- $\mathcal{F}'_{i;l}$ be the 0-dimensional \mathcal{O}_{C', p'_i} -module $\mathcal{O}_{C', p'_i} / (u_i^{n_{p'_i} l})$;
- $\hat{\mathcal{F}}'_{i;l}$ be the $\mathcal{O}_{\hat{C}'_i}$ -module associated to $\mathcal{F}'_{i;l}$;
- $\mathcal{F}_{i;l}$ be the \mathcal{O}_C -module $\rho_* \mathcal{F}'_{i;l}$;
- $\hat{\mathcal{F}}_{i;l}$ be the $\mathcal{O}_{\hat{C}}$ -module $\hat{\rho}_i * \hat{\mathcal{F}}'_{i;l} = \widehat{\rho_* \mathcal{F}'_{i;l}}$.

Then, one has the following lemma:

Lemma 2.7. [separation by punctual modules]. $length(\text{Supp}(\mathcal{F}_{i;l})) \geq l$ and $\text{Supp}(\hat{\mathcal{F}}_{i;l}) \subset \hat{C}_i$. In particular, if $l > b.i.i.(p)$, then $\mathcal{F}_{1;l}, \dots, \mathcal{F}_{k;l}$ are punctual 0-dimensional \mathcal{O}_C -modules at p that are non-isomorphic to each other.

Proof. As in the previous theme, we may assume that $u_i^{n_{p'_i}} = \rho^\#(f_i)$ for some $f_i \in \mathfrak{m}_p \subset \mathcal{O}_{C,p}$. Let $h \in \mathbb{C}[x]$ be a polynomial in one variable. Then, by construction, $h(u_i^{n_{p'_i}}) \cdot \mathcal{F}'_{i;l} = 0$ if and only if $h \in (x^l)$. In other words, $h(f_i) \cdot \mathcal{F}_{i;l} = 0$ if and only if $h \in (x^l)$. It follows that there exists a local section $m_{i;l}$ of $\mathcal{F}_{i;l}$ such that $f_i^{l-1} \cdot m_{i;l} \neq 0$. Consider the sub- \mathcal{O}_C -module $\mathcal{O}_C \cdot m_{i;l} \simeq \mathcal{O}_C / \text{Ann}(m_{i;l})$ of $\mathcal{F}_{i;l}$, where $\text{Ann}(m_{i;l})$ is the annihilator of $m_{i;l}$ in $\mathcal{O}_{C,p}$. Then,

$$m_{i;l}, f_i \cdot m_{i;l}, \dots, f_i^{l-1} \cdot m_{i;l}$$

are \mathbb{C} -linearly independent in $\mathcal{F}_{i;l}$, which implies that

$$1, f_i, \dots, f_i^{l-1}$$

are \mathbb{C} -linearly independent in $\mathcal{O}_{C,p}$. Since

$$\text{Span}_{\mathbb{C}}\{1, f_i, \dots, f_i^{l-1}\} \cap \text{Ann}(m_{i;l}) = 0$$

as \mathbb{C} -vector subspaces in $\mathcal{O}_{C,p}$, one has that $length(\text{Supp}(\mathcal{O}_{C,p} / \text{Ann}(m_{i;l}))) \geq l$ and, hence, that $length(\text{Supp}(\mathcal{F}_{i;l})) \geq l$. The rest of the lemma are immediate. \square

We say that $p'_1, \dots, p'_k \in \rho^{-1}(p) \subset C'$ are separated by the punctual \mathcal{O}_C -modules $\mathcal{F}_{1;l}, \dots, \mathcal{F}_{k;l}$ at $p \in C$ when $\mathcal{F}_{1;l}, \dots, \mathcal{F}_{k;l}$ as constructed above are non-isomorphic to each other.

Construction of embeddings $C' \hookrightarrow \mathfrak{M}_p^{0\text{Az}^f}(C)$ that descend to ρ .

We now proceed to prove Proposition 2.1 in three steps.

Step (a): Examination of a local model.

Consider the local ring $\mathcal{O}_{C' \times C', (p', p')} = \mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}$. (For simplicity of phrasing, here we use ‘=’ to mean ‘standard canonical isomorphism’.) Let $\mathfrak{m}_{p'} = (t_1) \subset \mathcal{O}_{C', p'}$ be the maximal ideal of the first factor and $\mathfrak{m}_{p'} = (t_2) \subset \mathcal{O}_{C', p'}$ be the maximal ideal of the second factor. Given $r \in \mathbb{N}$, compare the following two quotient $\mathcal{O}_{C' \times C', (p', p')}$ -modules:

$$M_1 := \frac{\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}}{((t_1 \otimes 1 - 1 \otimes t_2)^r, t_1 \otimes 1)} \quad \text{and} \quad M_2 := \frac{\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}}{((t_1 \otimes 1 - 1 \otimes t_2)^r, t_1^2 \otimes 1)}.$$

M_1 corresponds to the restriction of the $\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}$ -module $\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'} / (t_1 \otimes 1 - 1 \otimes t_2)^r$, which is flat over C' (the first factor), to over $p' \in C'$ (the first factor) while M_2 corresponds to the restriction of the same $\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}$ -module to over $v_{p'} \simeq \text{Spec}(\mathbb{C}[t_1]/(t_1^2)) \simeq \text{Spec}(\mathbb{C}[\varepsilon]) \subset C'$ (the first factor). They fit into an exact sequence, representing a class in $\text{Ext}_{C'}^1(M_1, M_1)$ (here $C' =$ the second factor),

$$0 \longrightarrow M_1 \xrightarrow{a} M_2 \xrightarrow{b} M_1 \longrightarrow 0$$

of $\mathbb{C}[\varepsilon]$ -modules with

$$\begin{aligned} M_1 &= \text{Span}_{\mathbb{C}} \{1 \otimes 1, 1 \otimes t_2^2, \dots, 1 \otimes t_2^{r-1}\}; \\ M_2 &= \text{Span}_{\mathbb{C}[\varepsilon]} \{1 \otimes 1, 1 \otimes t_2^2, \dots, 1 \otimes t_2^{r-1}\} \\ &= \text{Span}_{\mathbb{C}} \{1 \otimes 1, 1 \otimes t_2^2, \dots, 1 \otimes t_2^{r-1}, \varepsilon \otimes 1, \varepsilon \otimes t_2^2, \dots, \varepsilon \otimes t_2^{r-1}\}, \end{aligned}$$

where $a =$ multiplication by ε , and $b =$ quotient by εM_1 . As $\mathbb{C}[\varepsilon]$ -modules and with respect to the above bases (and with a vector identified as a column vector),

$$t_2 \text{ on } M_1 = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & 0 \\ & & & 1 & 0 \end{bmatrix}_{r \times r} \quad \text{and} \quad t_2 \text{ on } M_2 = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & 0 \\ & & & 1 & r\varepsilon \end{bmatrix}_{r \times r}.$$

Here all the missing entries in the $r \times r$ -matrices are 0. It follows that, as $\mathcal{O}_{C', p'}$ (the second factor) -modules,

$$\cdot t_2^l \cdot M_1 = 0 \text{ if and only if } l \geq r \quad \text{while} \quad t_2^l \cdot M_2 = 0 \text{ if and only if } l \geq r + 1.$$

In particular, the above short exact sequence (of $\mathcal{O}_{C', p'}$ -modules) doesn't split.

Step (b): Construction of a local embedding $C' \rightarrow \mathfrak{M}_{r_0}^{0\text{Az}^f}(C)$ that descend to ρ , for some $r_0 \in \mathbb{N}$.

Let

$$l_0 := 1 + \max\{b.i.i.(p) : p \in C_{\text{sing}}, C \text{ has multiple branches at } p\}$$

(by convention, $l_0 = 1$ if C has only single branch at each $p \in C_{\text{sing}}$) and

$$r_0 := l_0 \cdot l.c.m.\{n_{p'} : p' \in C'\} \in \mathbb{N}.$$

(Here, $l.c.m.$ = the ‘least common multiple’ in \mathbb{N} .) Since $n_{p'} = 1$ except for $\rho(p')$ in the finite set C_{sing} , r_0 is well-defined. Furthermore, since $\{n_{p'}\}_{\rho(p') \in C_{\text{sing}}}$ and $\{b.i.i.(p) : p \in C_{\text{sing}}\}$ (possibly

empty) depend only on the germ of C_{sing} in C , r_0 depends only on the germ of C_{sing} in C . Let $\tilde{\mathcal{E}}'$ be the $\mathcal{O}_{C' \times C'}$ -module

$$\tilde{\mathcal{E}}' = \mathcal{O}_{C' \times C'} / \mathcal{I}_{\Delta_{C'}}^{r_0}$$

and $\tilde{\mathcal{E}} := (id_{C'} \times \rho)_*(\tilde{\mathcal{E}}')$ on $C' \times C$. Then, it follows from the construction and Lemma 2.7 that the induced morphism

$$\tilde{\rho}_0 : C' \longrightarrow \mathfrak{M}_{r_0}^{0A_z^f}(C)$$

descends to ρ and sends distinct closed points of C' to distinct geometric points on $\mathfrak{M}_{r_0}^{0A_z^f}(C)$ (i.e. $\tilde{\rho}$ separates points of C'). Furthermore, it follows from the local study in Step (a) and Corollary 2.5 that all the extension classes $\alpha_{p'} \in \text{Ext}_C^1(\tilde{\mathcal{E}}_{p'}, \tilde{\mathcal{E}}_{p'})$, $p' \in C'$, $\tilde{\mathcal{E}}$ specifies are non-zero. This shows that $\tilde{\rho}_0$ separates also tangents of C' and hence is an embedding.

Step (c): Embeddings $C' \hookrightarrow \mathfrak{M}_r^{0A_z^f}(C)$ that descend to ρ , for all $r > r_0$.

Finally, to obtain an embedding $\tilde{\rho} : C' \rightarrow \mathfrak{M}_r^{0A_z^f}(C)$ for $r > r_0$ that descends to ρ , observe that the $\mathcal{O}_{C' \times C}$ -module $\mathcal{O}_{\Gamma_\rho}$ has the following properties:

- The corresponding extension class $\bar{\alpha}_{p'}$ in $\text{Ext}_C^1(\mathcal{O}_p, \mathcal{O}_p)$, where $p := \rho(p')$, vanishes if and only if $p \in C_{\text{sing}}$.

This implies that all the extension classes $\hat{\alpha}_{p'} \in \text{Ext}_C^1(\hat{\mathcal{E}}_{p'}, \hat{\mathcal{E}}_{p'})$, $p' \in C'$, as specified by the direct sum

$$\hat{\mathcal{E}} := \tilde{\mathcal{E}} \oplus \mathcal{O}_{\Gamma_\rho}^{\oplus(r-r_0)}$$

of $\mathcal{O}_{C' \times C}$ -modules, remain non-zero. Furthermore,

$$\text{Supp}((\tilde{\mathcal{E}} \oplus \mathcal{O}_{\Gamma_\rho}^{\oplus(r-r_0)})|_{p' \times C}) = \text{Supp}(\tilde{\mathcal{E}}_{p'}) \quad \text{for all } p' \in C'.$$

It follows that the morphism $\tilde{\rho} : C' \rightarrow \mathfrak{M}_r^{0A_z^f}(C)$ specified by $\hat{\mathcal{E}}$ on $C' \times C$ separates both points and tangents of C' and, hence, is an embedding that descends to ρ .

This concludes the proof of Proposition 2.1.

Remark 2.8. [non-uniqueness]. In general there can be other embeddings of C' into $\mathfrak{M}_r^{0A_z^f}(C)$ that descend also to ρ . Hence, the one constructed in the proof above is by no means unique.

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